Hunters Hill High School

Mathematics Extension 2

Trial Examination, 2016

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- The marks for each question are shown on the paper
- Show all necessary working in questions 11-16.

Total Marks: 100

Section I Pages 3-6

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7-14

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Section I

10 marks Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. ω is a non-real root of the equation $z^5 + 1 = 0$.

Which of the following is not a root of this equation?

(A) $\overline{\omega}$

(B) ω^2

(C) $\dfrac{1}{\omega}$

(D) ω^3

2. What is the acute angle between the asymptotes of the hyperbola $\dfrac{x^2}{3} - y^2 = 1$?

(A) $\dfrac{\pi}{6}$

(B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$

(D) $\dfrac{\pi}{2}$

3. What is the number of asymptotes on the graph of $f(x) = \dfrac{x^2}{x^2 - 1}$?

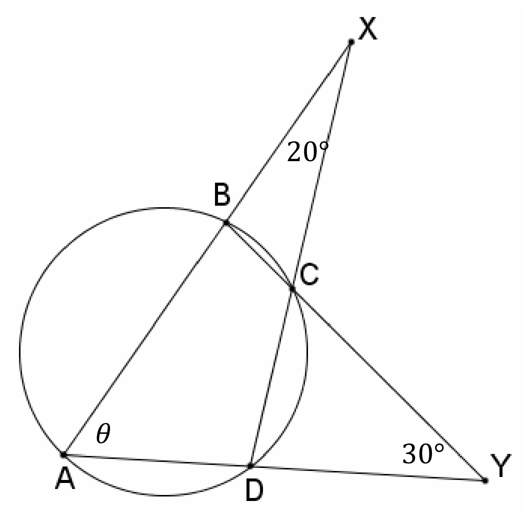
(A) 1

(B) 2

(C) 3

(D) 4

4. The size of the angle θ in the diagram below is:



(A) $50°$

(B) $55°$

(C) $60°$

(D) $65°$

5. The locus of the graph of $\arg\left(\dfrac{z-2}{z+2i}\right)=\dfrac{\pi}{2}$ is

(A) a semicircle passing through the origin

(B) a circle with centre at the origin

(C) an ellipse with a focus at the origin

(D) a hyperbola not passing through the origin

6. The equation of the tangent of the ellipse $x = 3\cos\theta$, $y = 2\sin\theta$ at the point where $\theta = \dfrac{\pi}{3}$ is:

(A) $6\sqrt{3}x - 4y - 5\sqrt{3} = 0$

(B) $2x - 3\sqrt{3}y - 12 = 0$

(C) $2x + 3\sqrt{3}y - 12 = 0$

(D) $6\sqrt{3}x + 4y - 5\sqrt{3} = 0$

7. The base of a solid is the circle $x^2 + y^2 = 4$. Every cross section of the solid taken perpendicular to the x-axis is a right-angled, isosceles triangle with its hypotenuse lying on the base of the solid.

Which of the following is an expression for the volume of the solid?

(A) $\frac{1}{4}\int_{-2}^{2}(4 - x^2)dx$

(B) $\int_{-2}^{2}(4 - x^2)dx$

(C) $2\int_{-2}^{2}(4 - x^2)dx$

(D) $4\int_{-2}^{2}(4 - x^2)dx$

8. $\int x \sin 2x \, dx =$

(A) $-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

(B) $-\frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$

(C) $\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

(D) $-2x\cos 2x + \sin 2x + C$

9. If $e^x + e^y = 1$, which of the following is an expression for $\frac{dy}{dx}$?

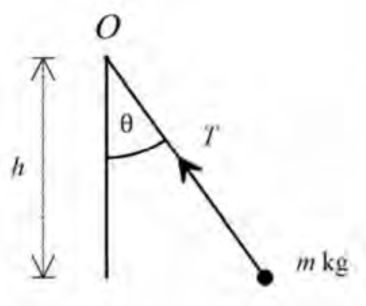
(A) $-e^{x-y}$

(B) e^{x-y}

(C) e^{y-x}

(D) $-e^{y-x}$

10. A particle of mass m moves in a horizontal circle with angular speed ω at a distance h below the point O.



Which of the following reflects the relationship between ω and h?

(A) $h = \omega^2 g$

(B) $h = \omega g$

(C) $g = \omega^2 h$

(D) $g = \omega h$

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hour and 45 minutes for this section
Begin each question on a NEW SHEET of paper.

In questions 11 – 16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

a. Factorise $z^2 + 2iz + 15$ **2**

b. Let $z = 1 + i$ and $u = 2 - i$. Find:

i. $\text{Im}(uz)$ **1**

ii. $|u - z|$ **1**

iii. $-i\bar{u}$ **1**

c. **i.** On an Argand diagram sketch the locus of z represented by $|z - 3| = 3$. **2**

ii. Explain why $\arg(z - 3) = 2 \arg z$ **1**

d. Find

i. $\int \sec^3 x \tan x \, dx$ **2**

ii. $\int_4^7 \frac{dx}{x^2 - 8x + 19}$ **3**

e. If $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that **2**

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a. Given the expression $\dfrac{6x^2+3x+1}{(x+1)(x^2+1)}$

i. Find numbers A, B and C such that **2**

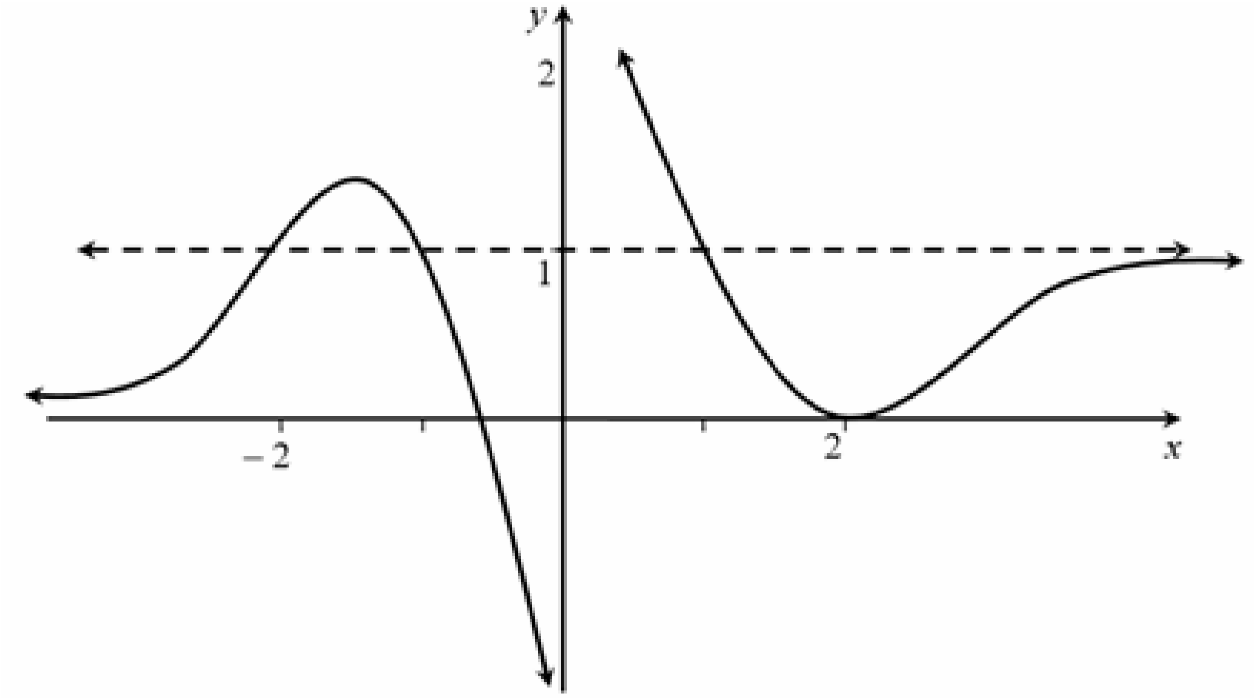
$$\frac{6x^2+3x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

ii. Find $\displaystyle\int \frac{6x^2+3x+1}{(x+1)(x^2+1)}\,dx$ **3**

b. i. Express $-\sqrt{3}-i$ in modulus-argument form. **2**

ii. Show that $\left(-\sqrt{3}-i\right)^6$ is a real number. **2**

c. The diagram below is a sketch of the function $y = f(x)$.
The lines $x = 0, y = 0$ and $y = 1$ are asymptotes.



Draw separate one-third page sketches of the following curves, clearly indicating any important features such as turning points or asymptotes.

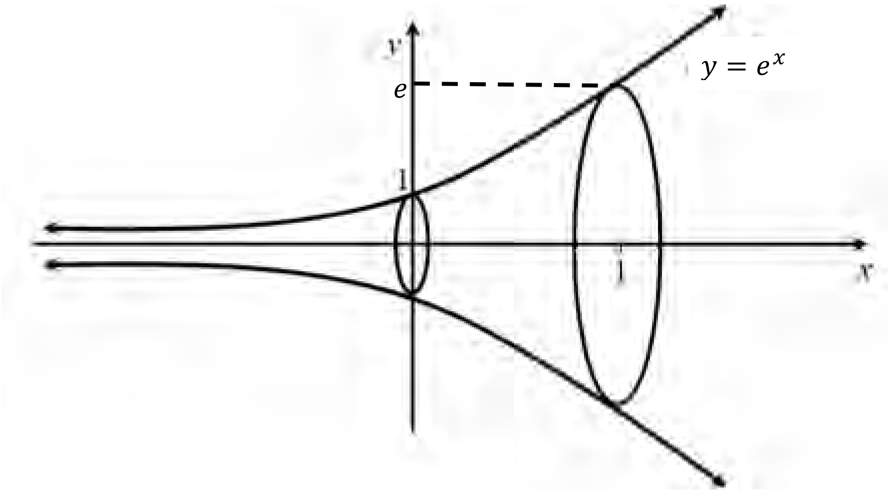
i. $y = \sqrt{f(x)}$ **3**

ii. $y = e^{f(x)}$ **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a. The arc defined by $y = e^x$, $0 \le x \le 1$, is rotated about the x-axis to form a curved bowl.



i. Using the method of cylindrical shells, show that the volume, V, of the solid that makes the bowl is given by **3**

$$V = \pi e^2 - 2\pi \int_1^e y \ln y \, dy$$

ii. Find the volume, leaving your answer in exact form. **3**

iii. Hence, evaluate **1**

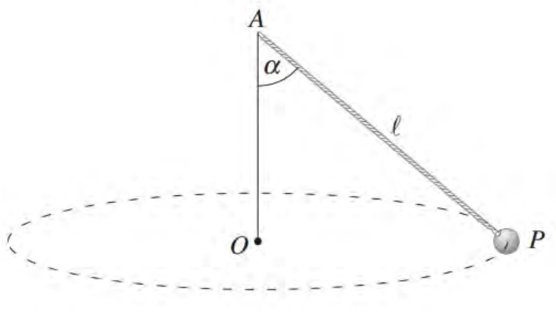
$$\int_0^1 e^{2x} dx$$

b. **i.** find all the 5th roots of -1 in modulus-argument form. **2**

ii. Sketch the 5th roots of -1 on an Argand diagram. **1**

Question 13 continued on next page

c. A particle P of mass m is attached by a string of length l to a point A. The particle moves with constant angular velocity ω in a horizontal circle with centre O which lies directly below A. The angle the string makes with OA is α.

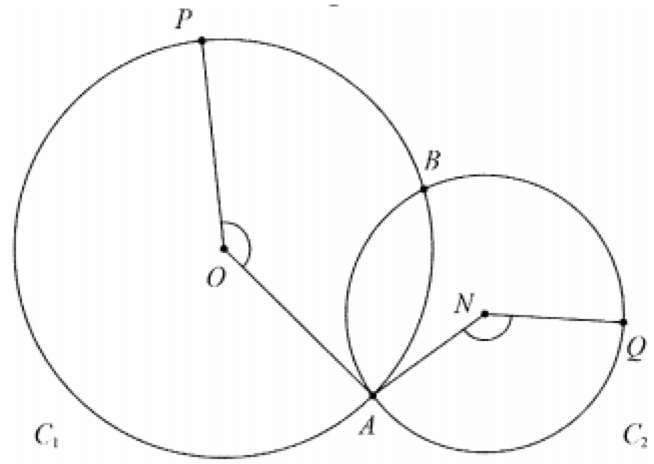


The forces acting on the particle are the tension, T, in the string and the force due to gravity, mg.

By resolving the forces acting on the particle in the horizontal and vertical directions, show that **3**

$$\omega^2 = \frac{g}{l\cos\alpha}$$

d. Two circle C_1 and C_2 with centres O and N respectively intersect at A and B. P lies on C_1 and Q lies on C_2 such that $\angle AOP = \angle ANQ$ and $\angle AOP > \angle AOB$, as shown in the diagram below. **3**



Prove that the points P, B and Q are collinear.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a. **i.** Show that $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx$. **2**

ii. Deduce that $\int_{-a}^{a} f(x)dx = \int_{0}^{a} \{f(x) + f(-x)\}dx$. **1**

iii. Show that $\sec^2 x - \sec x \tan x = \dfrac{1}{1+\sin x}$ **1**

iv. Hence, or otherwise, evaluate $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \dfrac{1}{1+\sin x}dx$. **2**

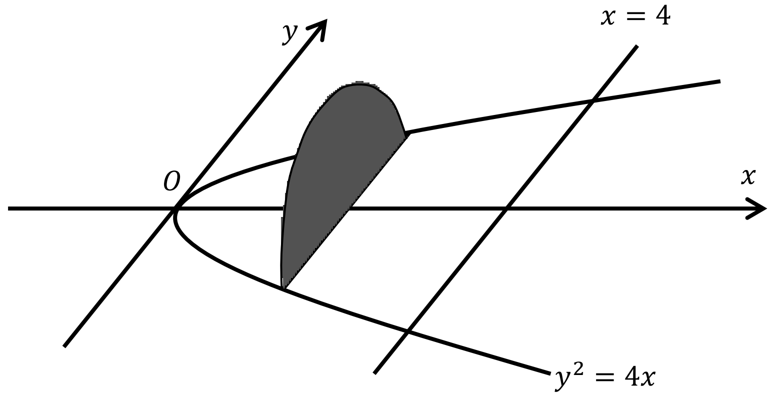
b. **i.** Prove that $a^2 + b^2 \geq 2ab$ **1**

ii. Hence, or otherwise, prove that $(p+2)(q+2)(p+q) \geq 16pq$ where p and q are positive real numbers. **2**

c. **i.** Express $(5-i)^2(1+i)$ in the form $a + ib$ where a and b are real. **1**

ii. Hence, prove that $\tan^{-1}\dfrac{7}{17} + 2\tan^{-1}\dfrac{1}{5} = \dfrac{\pi}{4}$. **2**

d. The base of a solid is the region in the xy-plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$. Each cross section perpendicular to the x-axis is a semicircle.

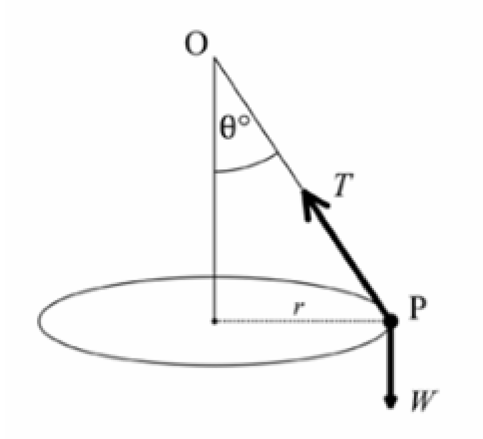


Find the volume of the solid. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

a. A body P of mass 0.5kg is suspended from a fixed point O by means of a light rod of length 1 metre.
The mass is rotated in a horizontal circle at a constant speed v ms^{-1}.
The rod makes an angle θ with the downward vertical direction as shown in the diagram below.



The tension in the rod is T newtons and the weight of P is W newtons.
The radius of the circle is r metres.

Assume $g = 9.8$ ms^{-2} and $\theta = 30°$.

i. Show that $\tan\theta = \dfrac{v^2}{rg}$. **3**

ii. Find the tension T. **1**

iii. Find the speed v ms^{-1} of P. **1**

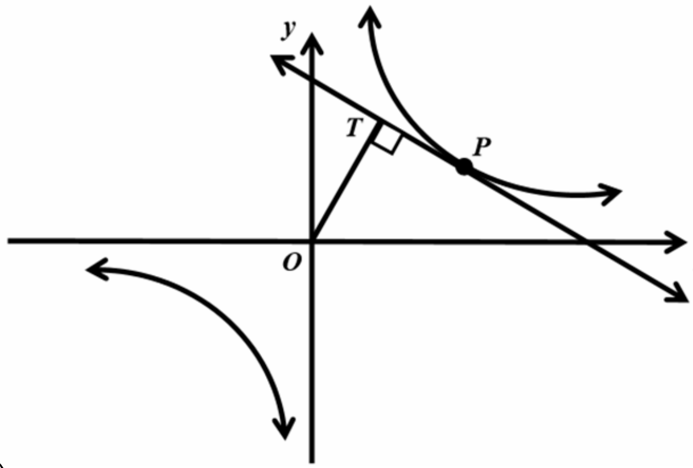
iv. Find the period of the motion. **1**

b. Using the substitution $t = \tan\dfrac{x}{2}$, evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}.$$

Question 15 continued on next page

c.



The point $P\left(cp, \frac{c}{p}\right)$ lies on the hyperbola $xy = c^2$. The point T lies at the foot of the perpendicular drawn from the origin O to the tangent at P.

i. Show that the tangent at P has equation $x + p^2y = 2cp$. **2**

ii. If the coordinates of T are (x_1, y_1)show that $y_1 = p^2x_1$. **1**

iii. Show that the locus of T is given by $(x^2 + y^2)^2 = 4c^2xy$. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

a. Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2}\,dx$.

i. Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \geq 1$. **2**

ii. Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$ and $I_1 = \frac{1}{2} - \frac{1}{e}$. **1**

iii. Prove by mathematical induction that for all $n \geq 1$, **3**

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2eI_n}{n!}$$

iv. It is given that $0 \leq I_n \leq 1$ because $0 \leq x^{2n+1}e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.
[Do NOT prove this]

Use this fact to help evaluate $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$, giving your answer in exact form. **1**

b. An object on the surface of a liquid is released at time $t = 0$ and immediately sinks. Let x be its displacement in metres in a downward direction from the surface at time t seconds.

The equation of motion is given by

$$\frac{dv}{dt} = 10 - \frac{v^2}{40},$$

where v is the velocity of the object.

i. Show that $v = \frac{20(e^t - 1)}{e^t + 1}$. **4**

ii. Use $\frac{dv}{dt} = v\frac{dv}{dx}$ to show that $x = 20\log_e\left(\frac{400}{400 - v^2}\right)$. **2**

iii. How far does the object sink in the first 4 seconds? **2**

End of paper

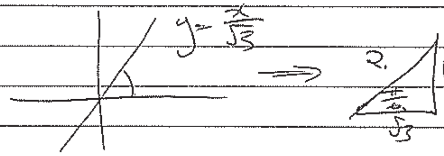
MXX – 2016 TRIAL – SOLUTIONS

SECTION I

1. C

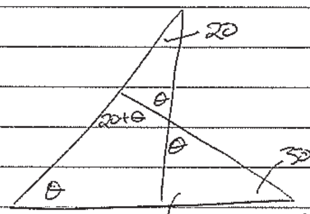
if ω is a 5^{th} root, then $\omega^2, \omega^3, \omega^4, \omega^5$ are.
also, roots of unity occur in conjugate pairs

2. C



3. C

4. D



$180-(30+\theta) = 20+\theta$

$2\theta = 130$

5. A

6. C

$\frac{dx}{d\theta} = -3\sin\theta, \quad \frac{dy}{d\theta} = 2\cos\theta$

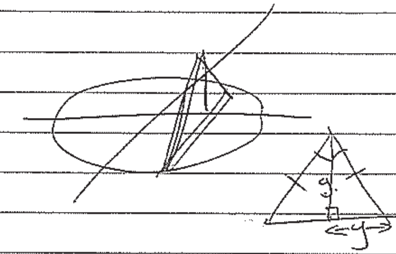
$\frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta} \quad \left(= \frac{dy}{d\theta} \times \frac{d\theta}{dx}\right)$

$= -\frac{2}{3} \cdot \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \frac{-2}{3\sqrt{3}}$

then point-gradient

7. B.



$A = \frac{1}{2}(2y) \cdot y$

$= y^2$

$= 4 - x^2$

$\delta V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} (4 - x^2)\delta x$

8. A.

$\int x \sin 2x\,dx = x\left(\frac{-\cos 2x}{2}\right) - \int \left(\frac{-\cos 2x}{2}\right) dx$

9. A.

$\frac{d}{dx}: \quad e^x + e^y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{e^x}{e^y}$

$= -e^x \cdot e^{-y}$

$= -e^{x-y}$

10. C.

$h = \frac{g}{\omega^2}$

$\omega^2 h = g$

(11)

a. $z^2+2iz+15 = z^2+2iz-15i^2$ 2.

$= (z+5i)(z-3i)$

b. $z=1+i$, $u=2-i$

i $\text{Im}(uz) = \text{Im}((1+i)(2-i))$ 1.

$= \text{Im}(2+2i-i+1)$

$= \text{Im}(3+i)$

$= 1$

ii. $|u-z| = |2-i-(1+i)|$ 1.

$= |2-i-1-i|$

$= |1-2i|$

$= \sqrt{1^2+(2)^2}$

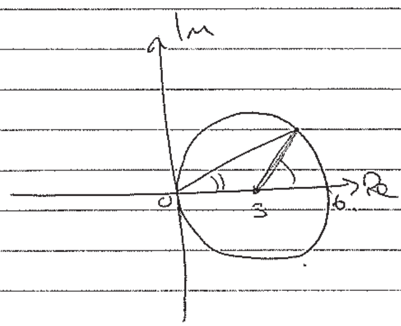
$= \sqrt{5}$

iii. $-i\bar{u} = -i(2+i)$ 1.

$= -2i+1$

$= 1-2i.$

c. i.



ii Angle at centre is twice angle at circumference subtended by same arc

d i. $I = \int \sec^2 x \cdot \sec x \cdot \tan x \, dx$

let $u = \sec x$

$du = \sec x \cdot \tan x \cdot dx$

$I = \int u^2 \, du$

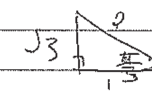
$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}\sec^3 x + c.$

ii. $\int_4^7 \frac{dx}{x^2-8x+19} = \int_4^7 \frac{dx}{x^2-8x+16+3}$

$= \int_4^7 \frac{dx}{(x-4)^2+3}$

$= \left[\frac{1}{\sqrt{3}} \tan^{-1}\frac{x-4}{\sqrt{3}}\right]_4^7$



$= \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{7-4}{\sqrt{3}} - \tan^{-1}\frac{4-4}{\sqrt{3}}\right)$

$= \frac{\pi}{3\sqrt{3}}.$

=

e. $z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta).$

$z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$

Q2.

a. i. $\dfrac{6x^2+3x+1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ 2.

$6x^2+3x+1 = A(x^2+1) + (Bx+C)(x+1)$

$= x^2(A+B) + x(B+C) + (A+C)$

comparing coefficients

$A + B = 6$ — i

$B + C = 3$ — ii

$A + C = 1$ — iii

i − iii

$B - C = 5$ — iv

ii + iv $2B = 8$

$B = 4$.

$\therefore C = -1$

$A = 2$

$\therefore A=2, B=4, C=-1$

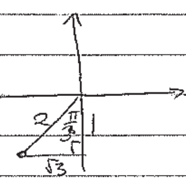
ii. $\displaystyle\int \frac{6x^2+3x+1}{(x+1)(x^2+1)}\,dx = \int\left(\frac{2}{x+1} + \frac{4x-1}{x^2+1}\right)dx$ 2.

$\displaystyle = \int\left(\frac{2}{x+1} + \frac{4x}{x^2+1} - \frac{1}{x^2+1}\right)dx$

$= 2\ln(x+1) + 2\ln(x^2+1) - \tan^{-1}x + c$

b i. $\sqrt{3} - i$ 2.

$= 2\text{cis}\left(-\frac{5\pi}{6}\right)$



ii. $(-\sqrt{3}-i)^6 = 2^6 \text{ cis } 6\left(-\frac{5\pi}{6}\right)$

$= 2^6 \text{cis}(-5\pi)$

$= 2^6[\cos(-5\pi) + i\sin(-5\pi)]$

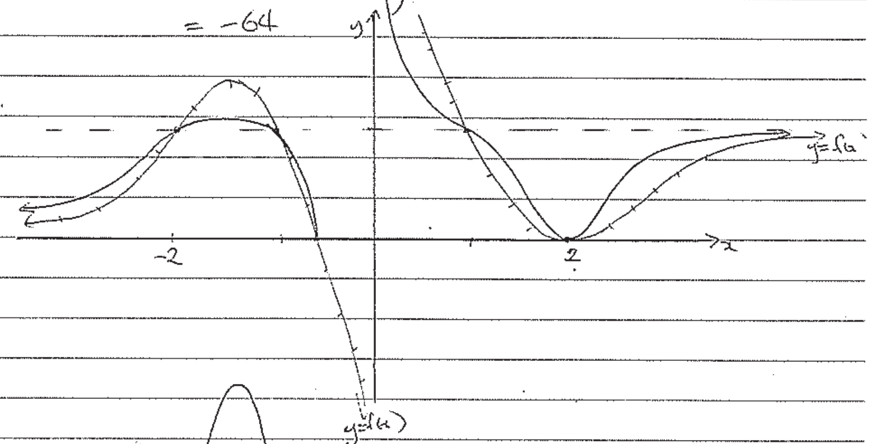
$= ~~2^6(~~$

$= 2^6(\cos(-\pi) + i\sin(-\pi))$

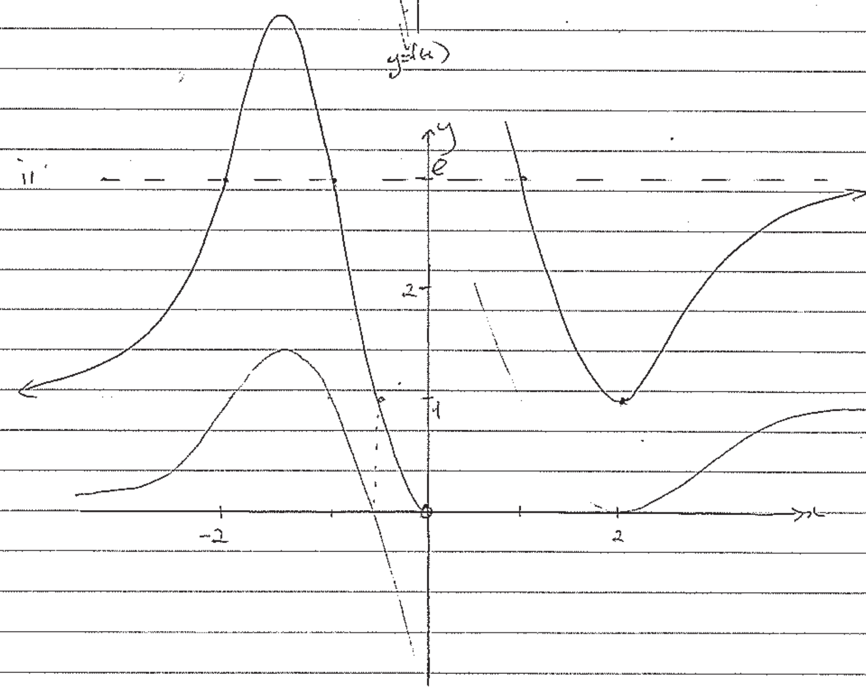
$= 2^6(-1 - 0i)$

$= -64$

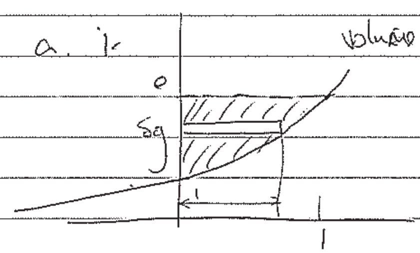
c i.



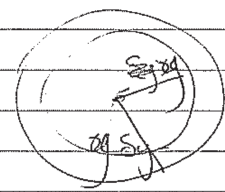
ii



Q13. a. i.



volume of bowl = cylinder − $\int$ shaded



Shaded Volume

$\delta A = \pi((y+\delta y)^2 - y^2)$

$= \pi(y^2 + 2y\delta y + \delta y^2 - y^2)$

$= 2y\pi\delta y$

height(x) = $\ln y$

$\delta V = 2\pi y \delta y . \ln y$

$V = \lim_{\delta y \to 0} \sum_{y=1}^{e} 2\pi y \ln y \, \delta y$

cylinder $= \pi . e^2 . 1$

$= 2\pi \int_1^e y \ln y \, dy.$

∴ Volume of bowl, $V = \pi e^2 - 2\pi \int_1^e y \ln y \, dy$

ii $V = 2\pi \int_1^e y \ln y \, dy = 2\pi \left(\left[\frac{y^2}{2} . \ln y \right]_1^e - \int_1^e \frac{y^2}{2} . \frac{1}{y} \, dy \right)$

$du = y \, dy$

$dv = \ln y \, dy$

$= 2\pi \left(\frac{e^2}{2} \ln e - \frac{1^2}{1} . \ln 1 - \left[\frac{y^2}{4} \right]_1^e \right)$

$= 2\pi \left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \right)$

$= \frac{\pi}{2}(1 + e^2)$

$V = \pi e^2 - \frac{\pi}{2}(1 + e^2)$

$= \frac{\pi e^2}{2} - \frac{\pi}{2}$

$= \frac{\pi}{2}(e^2 - 1)$ units^3

iii) $\int_0^1 e^{2x} dx.$

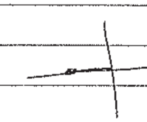
Volume of bowl also equal to $\pi \int_0^1 (e^x)^2 dx.$
when rotated about x-axis, slices perpendicular to axis

$\therefore \pi \int_0^1 e^{2x} dx = \frac{\pi}{2}(e^2 - 1)$

$\therefore \int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - 1)$

$= 3.194528049$

b. i $z^5 = -1$



$z^5 = r^5 \text{cis} 5\theta$

$-1 = \text{cis} \, \pi$

equating

$r^5 = 1$

$\therefore r = 1.$

$\text{cis} 5\theta = \text{cis} \, \pi$

$5\theta = \pi + 2k\pi$

$\theta = \frac{\pi(2k+1)}{5}$

for $k=0$, $z_1 = \text{cis} \frac{\pi}{5}$

$k=2$ $z_2 = \text{cis} \, \pi$

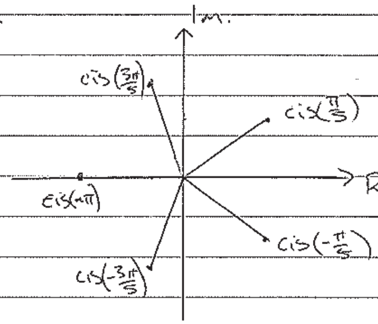
$k=3$ $z_3 = \text{cis} \frac{7\pi}{5}$

$= \text{cis}\left(-\frac{3\pi}{5}\right)$

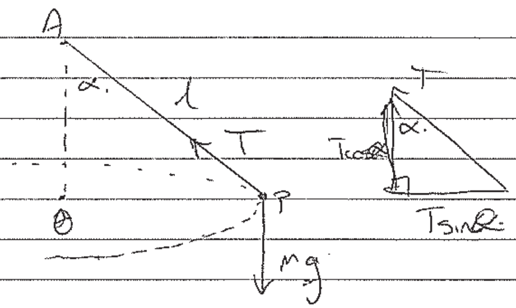
$k = 1, \quad z_4 = \text{cis}\,\frac{3\pi}{5}$

$k = -1, \quad z_5 = \text{cis}\left(-\frac{\pi}{5}\right)$

ii



c.



vertically: $T\cos\alpha = mg$ — i

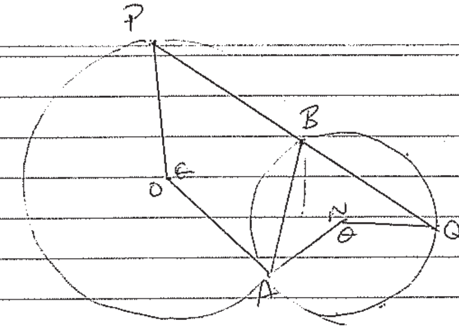
radially: $T\sin\alpha = mr\omega^2$ — ii

ii ÷ i $\quad \frac{T\sin\alpha}{T\cos\alpha} = \frac{mr\omega^2}{mg}$

$\frac{r}{l\cos\alpha} = \frac{r\omega^2}{g} \qquad \sin\alpha = \frac{r}{l}$

$\therefore \omega^2 = \frac{g}{l\cos\alpha}$

d.



$\angle AOP = \angle ANQ$ (given)
$= \theta$

$\angle PBA = (\text{obtuse } \angle AOP) \times \frac{1}{2}$ (angle at centre twice angle at circumference)
$= (2\pi - \theta) \times \frac{1}{2}$
$= \pi - \frac{\theta}{2}$

$\angle QBA = \angle QNA \times \frac{1}{2}$ (angle at centre twice angle at circumference)
$= \frac{\theta}{2}$

$\angle PBA + \angle QBA = \pi - \frac{\theta}{2} + \frac{\theta}{2}$

$= \pi$

$\therefore$ P, B and Q are collinear.

Q14.

a. i. $LHS = \int_{-a}^{0} f(x)\,dx$ 2.

$= \Big[F(x)\Big]_{-a}^{0}$

$= F(0) - F(-a)$

$RHS = \int_{0}^{a} f(-x)\,dx$

$= \Big[-F(-x)\Big]_{0}^{a}$

$= -F(-a) - (-F(0))$

$= F(0) - F(-a)$

$= LHS$

ii. $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ 1.

$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{a} \{f(x) + f(-x)\}\,dx$

iii. $LHS = \sec^2 x - \sec\tan x = \frac{1}{1+\sin}$ 1.

$= \frac{1}{\cos^2 x} - \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$

$= \frac{1 - \sin x}{\cos^2 x}$

$= \frac{1 - \sin x}{1 - \sin^2 x}$

$= \frac{1 - \sin x}{(1+\sin x)(1 - \sin x)}$

$= \frac{1}{1+\sin x} = RHS$

iv. $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{1+\sin x}\,dx = \int_{0}^{\frac{\pi}{6}} \left\{\frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)}\right\}dx$ 2

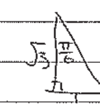
$= \int_{0}^{\frac{\pi}{6}} \{\sec^2 x - \sec x\tan x + \sec^2(-x) - \sec(-x)\tan(-x)\}$

$= \int_{0}^{\frac{\pi}{6}} \{\sec^2 x - \sec x\tan x + \sec^2 x - \sec x(-\tan x)\}\,dx$

$= \int_{0}^{\frac{\pi}{6}} 2\sec^2 x\,dx$

$= 2\Big[\tan x\Big]_{0}^{\frac{\pi}{6}}$

$= 2\left(\tan\frac{\pi}{6} - \tan 0\right)$



$= 2\cdot\left(\frac{1}{\sqrt{3}} - 0\right)$

$= \frac{2}{\sqrt{3}}$

b. i. $(a-b)^2 \geq 0$ 1.

$a^2 - 2ab + b^2 \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

ii. $p+2 = (\sqrt{p})^2 + (\sqrt{2})^2 \geq 2\sqrt{p}\sqrt{2}$. 2.

$q+2 = (\sqrt{q})^2 + (\sqrt{2})^2 \geq 2\sqrt{q}\sqrt{2}$.

$p+q \geq 2\sqrt{pq}$.

$\therefore (p+2)(q+2)(p+q) \geq 2\sqrt{p}\sqrt{2}\cdot 2\sqrt{q}\cdot\sqrt{2}\cdot 2\sqrt{p}\sqrt{q}$

$\geq 16pq$

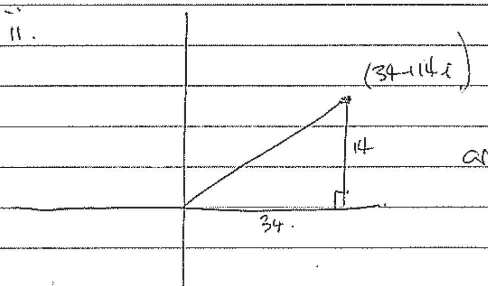
c. i. $(5-i)^2(1+i) = (25-10i+(-1))(1+i)$

$= (24-10i)(1+i)$

$= 24-10i+24i+10$

$= 34+14i$

ii.



$\arg(34+14i) = \tan^{-1}\frac{14}{34}$

$= \tan^{-1}\frac{7}{17}$

$\arg(zw) = \arg(z) + \arg(w)$

$\therefore \arg(34+14i) = \arg(24-10i) + \arg(1+i)$

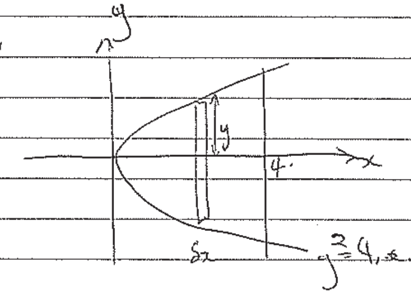
$= 2\arg(5-i) + \arg(1+i)$

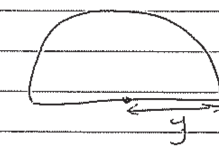
$\tan^{-1}\frac{7}{17} = 2\tan^{-1}\left(\frac{-1}{5}\right) + \tan^{-1}\frac{1}{1}$

$= -2\tan^{-1}\frac{1}{5} + \frac{\pi}{4}$

$\therefore \tan^{-1}\frac{7}{17} + 2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$

d.





$\delta V = \frac{\pi}{2} y^2 \delta x$

$= \frac{\pi}{2} 4x \delta x$

$= 2\pi x \delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{4} 2\pi x \delta x$

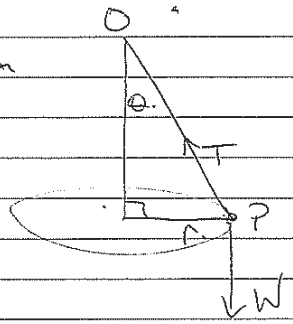
$= 2\pi \int_0^4 x \, dx$

$= 2\pi \left[\frac{x^2}{2}\right]_0^4$

$= 2\pi\left(\frac{4^2}{2} - \frac{0^2}{2}\right)$

$= 16\pi \text{ units}^3$

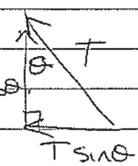
Q15 a



$m = 0.5kg$
$OP = 1m$
$W = mg.$

i.

vertically: $T\cos\theta = mg$ ① ÷cosθ



radially: $T\sin\theta = \frac{mv^2}{r}$ ②

② ÷ ① $\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2}{mrg}$

$\therefore \tan\theta = \frac{v^2}{rg}$

ii. from ①

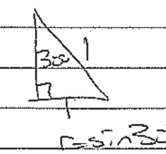
$T\cos\theta = mg$

$T = \frac{mg}{\cos\theta}$

$= \frac{0.5 \times 9.8}{\cos 30°}$

$= 5.658032638$ Newtons

iii. $\tan\theta = \frac{v^2}{rg}$



$v^2 = rg\tan\theta$

$= \sin 30° \times 9.8 \times \tan 30°$

$= \frac{49}{10\sqrt{3}}$

$v = 1.68196799\ ms^{-1}$

iv. $\omega = \frac{v}{r}$

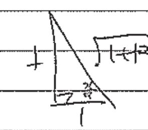
period $= \frac{2\pi}{\omega}$

$= \frac{2\pi r}{v}$

$= \frac{2\pi \sin 30°}{1.68}$

$= 1.867807635$ s.

b. $t = \tan\frac{x}{2}$



$\sin x = \frac{2t}{1+t^2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx.$

$dx = 2dt\,\cos^2\frac{x}{2}$

$= 2dt \cdot \frac{1}{1+t^2}$

$= \frac{2dt}{1+t^2}$

also, when

$x = 0,\ t = \tan 0$
$= 0$

$x = \frac{\pi}{2},\ t = \tan\frac{\pi}{4}$
$= 1$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{1+t^2}{1+t^2+2t} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{(1+t)^2}$$

$$= 2\left[\frac{(1+t)^{-1}}{-1}\right]_0^1$$

$$= 2\left(-\frac{1}{1+1} - \left(-\frac{1}{1+0}\right)\right)$$

$$= 2\left(-\frac{1}{2} + 1\right)$$

$$= 1$$

c. i. $xy = c^2$

$\frac{d}{dx}$: $1 \cdot y + x\frac{dy}{dx} = 0$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

at P $m = -\frac{\frac{c}{p}}{cp}$

$$= -\frac{1}{p^2}$$

by point-gradient

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$p^2 y - px = -x + cp$$

$$\therefore \; x + p^2 y = 2cp.$$

ii. $OT \perp PT$

so, gradient of OT is p^2 $(m_1 m_2 = -1)$

passing through $(0,0)$

by point-gradient.

$$y - 0 = p^2(x - 0)$$

$$y = p^2 x$$

as T satisfies line

$$y_1 = p^2 x_1$$

iii $x + p^2 y = 2cp$

also $p^2 = \frac{y}{x}$

so,

$$x + \frac{y}{x} \cdot y = 2c\sqrt{\frac{y}{x}}$$

$$x^2 + y^2 = 2c\sqrt{xy}$$

$$(x^2 + y^2)^2 = 4c^2 xy$$

16 a. a. $I_n = \int_0^1 x^{2n+1} e^{-x^2} dx$ 2.

i. $= \int_0^1 x^{2n} . x e^{-x^2} dx$

let $u = x^{2n}$ $du = 2n x^{2n-1} dx$

$dv = x e^{-x^2} dx$ $v = -\frac{1}{2} e^{-x^2}$

$$I_n = \left[x^{2n} \left(-\frac{1}{2}\right) e^{-x^2} \right]_0^1 - \frac{2n}{(-2)} \int_0^1 x^{2n-1} . e^{-x^2} dx$$

$$= \left(-1^{2n} . \frac{1}{2} e^{-1} - 0^{2n} . \frac{1}{2} e^0 \right) + n \int_0^1 x^{2(n-1)+1} . e^{-x^2} dx$$

$$= -\frac{1}{2e} + n I_{n-1}$$

ii. $I_0 = -\frac{1}{2} \int_0^1 -2x . e^{-x^2} dx$

$$= -\frac{1}{2} \left[e^{-x^2} \right]_0^1$$

$$= -\frac{1}{2} \left(e^{-1} - e^{-0} \right)$$

$$= \frac{-1}{2e} + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{2e}$$

$\therefore$ $I_1 = -\frac{1}{2e} + (1) I_0$

$$= -\frac{1}{2e} + \frac{1}{2} - \frac{1}{2e}$$

$$= \frac{1}{2} - \frac{1}{e}.$$

iii. Prove true for $n = 1$

LHS $= 1 + \frac{1}{1!}$ RHS $= e - \frac{2e I_1}{1!}$

$= 2$ $= e - 2e \left(\frac{1}{2} - \frac{1}{e} \right)$

$= e - \frac{2e}{2} + \frac{2e}{e}$

$= 2$

$=$ LHS.

$\therefore$ true for $n = 1$

Assume true for $n = k$.

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} = e - \frac{2e I_k}{k!}$$

Prove true for $n = k+1$

i.e. $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \frac{1}{(k+1)!} = e - \frac{2e I_{k+1}}{(k+1)!}$

$$\text{LHS} = e - \frac{2e I_k}{k!} + \frac{1}{(k+1)!}$$

$$= e - \frac{2e}{k!} I_k \frac{(k+1)}{k+1} + \frac{1}{(k+1)!} . \frac{2e}{2e}.$$

$$= e + \frac{2e}{(k+1)!} \left(-(k+1) I_k + \frac{1}{2e} \right)$$ but $I_{k+1} = \frac{-1}{2e} + (k+1) I_k$

$$= e + \frac{2e}{(k+1)!} \left(-I_{k+1} \right)$$

$$= e - \frac{2e I_{k+1}}{(k+1)!}$$

$=$ RHS

$\therefore$ by induction, this is true.

10. $0 \leq I_n \leq 1$

as $n \to \infty$, $\frac{2e I_n}{n!} \to 0$

$\therefore 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e.$

and $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e - 1$

b. i. $\frac{dv}{dt} = 10 - \frac{v^2}{40}$

$= \frac{400 - v^2}{40}$

$\int \frac{40\,dv}{400 - v^2} = \int dt$

$\frac{40}{(20-v)(20+v)} = \frac{A}{20-v} + \frac{B}{20+v}$

$40 = A(20+v) + B(20-v)$

at $v = 20$,

$40 = A(20+20) + B(0)$

$A = 1$

at $v = -20$

$40 = A(0) + B(20-(-20))$

$B = 1$

$\int_0^v \left(\frac{1}{20-v} + \frac{1}{20+v}\right) dv = \int_0^t dt$

$\left[-\ln(20-v) + \ln(20+v)\right]_0^v = \left[t\right]_0^t$

$-\ln(20-v) + \ln(20+v) - (-\ln(20-0) + \ln(20+0)) = t - 0$

$\ln \frac{(20+v) \cdot 20}{(20-v) \cdot 20} = t.$

$\frac{20+v}{20-v} = e^t.$

$20 + v = e^t(20-v)$

$= 20e^t - ve^t.$

$ve^t + v = 20e^t - 20$

$v(e^t + 1) = 20(e^t - 1)$

$v = \frac{20(e^t - 1)}{e^t + 1}$

ii. $v\frac{dv}{dx} = 10 - \frac{v^2}{40}$

$\int_0^v \frac{40v\,dv}{400 - v^2} = \int_0^x dx$

$\left[-20\ln(400 - v^2)\right]_0^v = x$

$x = -20\ln(400 - v^2) + 20\ln(400 - 0)$

$= 20\ln\left(\frac{400}{400 - v^2}\right)$

iii at $t=4$

$$v = \frac{20(e^4-1)}{e^4+1}$$

$= 19.2805516$

$$x = 20\ln\left(\frac{400}{400-19.28^2}\right)$$

$= 5.300010989$

$\therefore$ object sinks 5.3 m